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Mean first-passage and residence times of random walks on asymmetric disordered chains

Pedro A Pury[1] **and Manuel O Cáceres**[2,3]

[1] Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina
[2] Centro Atómico Bariloche and Instituto Balseiro, 8400 San Carlos de Bariloche, Río Negro, Argentina
[3] Consejo Nacional de Investigaciones Científicasy Técnicas, Argentina

E-mail: pury@famaf.unc.edu.ar

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Abstract
An algebraic derivation is presented which yields the exact solution of the mean first-passage and mean residence times of a one-dimensional asymmetric random walk for quenched disorder. Two models of disorder are analytically treated. Both absorbing–absorbing and reflecting–absorbing boundaries are considered. Particularly, the interplay between asymmetry and disorder is studied.

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1. Introduction

The problems of the first-passage time (FPT) [1, 2] and the residence time (RT) [3] are very important issues in random walk theory. Moreover, several properties of diffusion and transport in disordered systems are based on these concepts. Here, we do not want to present a survey of the enormous literature in the field of mean first-passage time (MFPT) and mean residence time (MRT) of random walks, nevertheless, we wish to single out the works involved with analytical or exact results, mainly in one-dimensional disordered systems. Goldhirsch and Gefen [4] developed an analytical method for calculating MFPT for branched networks such as finite segments with dangling bonds and loops. The method is based on the generating function and was generalized for biased walks [5]. Extensions of the generating function method were done for analysing the probability distribution function of FPT [6] and the current autocorrelation function [7]. Later on, the generating function method was used for random one-dimensional chains [8, 9], particularly for the Sinai problem [10]. Explicit expressions for the MFPT, in terms of the basic jump probabilities for a discrete time random walk with a reflecting boundary, were obtained independently by Van den Broeck [11], Le Doussal [12] and Murthy and Kehr [13] by different methods. It is interesting to remark that Gardiner [14] had previously reported explicit MFPT formulae. An exact solution of the generating function for the first-passage probability was presented by Raykin [15] using enumerative combinatorics for summing up over trajectories of the random walker [16]. The distribution of escape probabilities was computed exactly by Sire [17] and the exact renormalization group analysis was performed by Le Dousal *et al* [18]. In the last few years, one of the main applications of the exact expressions for MFPT and MRT in one-dimensional lattices was the study of exciton migration in treelike dendrimers (light harvesting antennae) [19, 20].

A successful perturbative theory for survival statistics in disordered chains is the finite effective medium approximation (FEMA) [21]. Extensions of FEMA to biased media [22] and periodically forced boundary conditions [23] were presented. A unified framework for the FPT and RT statistics in finite disordered chains with bias was also presented by the authors in [24], where exact equations for the quantities averaged over disorder were obtained for both problems and its solutions up to first order in the bias parameter were constructed retaining full dependence on the system's size and the initial condition.

In this paper, we obtain explicit analytical expressions for the MFPT and MRT of random walks on a one-dimensional lattice for a quenched realization of disorder. Then, we consider two models for the disorder in the hopping transitions and average the expressions, in each case, on the realizations of disorder. The outline of the paper is as follows. The starting point of our formulation is given in section 2. In sections 2.1 and 2.2 we discuss the algebraic method that allows us to obtain the exact dependence of the MFPT and the MRT on the set of transition probabilities. The biased nondisordered chain is treated in section 3, whereas two models of disorder with asymmetric transition probabilities are analysed in sections 4 and 5. The effects on MFPT and MRT of cutting the chain, at a reflecting site, are considered in section 6. Finally, in section 7, we briefly summarize the principal results of our study.

2. Survival and residence probabilities

We start considering a random walk on a discrete one-dimensional lattice with nearest neighbour hopping; jumping from site n to $n+1$ with transition probability w_n^+, or to site $n-1$ with transition probability w_n^-. In this manner, the walker has a sojourn probability $1 - w_n^+ - w_n^-$ per unit time at site n. The conditional probability, $P(m,t|n)$, of finding the walker at site m at time t, given that it was initially at site n, satisfies a Markovian master equation for a given realization of the set $\{w_j^\pm\}$.

We are concerned with the survival and residence probabilities in the finite interval $D = [-M, L]$ on the chain. The first is the probability, $S_n(t)$, of remaining in D at time t, without exiting, if the walker initially began at site $n \in D$. The second is the probability, $R_n(t)$, of finding the particle within the domain D at time t, given that it was initially at site n (not necessarily in D). The dynamical evolution of both quantities follows from the backward master equation [24]

$$\partial_t F_n(t) = w_n^+(F_{n+1}(t) - F_n(t)) + w_n^-(F_{n-1}(t) - F_n(t)) \tag{2.1}$$

where

$$F_n(t) = \sum_{m\in D} P(m,t|n). \tag{2.2}$$

The survival probability is the solution of equation (2.1) with the initial condition $S_n(t=0) = 1$, for all $n \in D$, considering the boundary conditions $S_{-(M+1)}(t) = S_{L+1}(t) = 0$, for all t. On the other hand, the residence probability is the solution of equation (2.1) with the

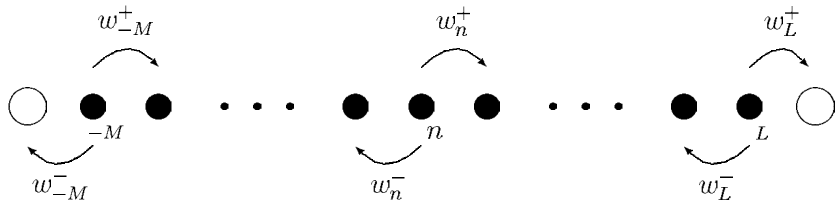

Figure 1. Graphical illustration of the definition of the probability transitions. The boundary conditions for the first-passage time problem are perfect tramps (○) in the segment extremes.

initial condition $R_n(t=0) = 1$ if $n \in D$, or 0 otherwise, fulfilling the boundary condition $R_n(t) \to 0$, for $|n| \to \infty$ for all finite t.

Finally, MFPT and MRT can be obtained from the asymptotic limit of the Laplace transformed (denoted by hats) survival and residence probabilities, respectively [24],

$$T_n = \lim_{s\to 0} \hat{S}_n(s) \tag{2.3a}$$

$$\tau_n = \lim_{s\to 0} \hat{R}_n(s). \tag{2.3b}$$

2.1. Mean first-passage time

In the first-passage time problem the random walker only jumps inside a finite interval with absorbing ends, as shown in figure 1. The boundary can be simply modelled by setting $w^+_{-(M+1)} = w^-_{L+1} = 0$ (i.e. the walker cannot jump back into the interval once it has been tramped on $-(M+1)$ or $L+1$). From the Laplace transform of the evolution equation (2.1), using that $S_n(0) = 1$ for all $n \in D$, and taking the limit of equation (2.3a), we obtain the corresponding equation for the MFPT

$$w^+_n(T_{n+1} - T_n) + w^-_n(T_{n-1} - T_n) = -1 \qquad \forall n \in D. \tag{2.4}$$

This equation must be supplemented with the conditions: $T_{-(M+1)} = T_{L+1} = 0$. Equation (2.4) is a three-term recursion formula. To get a two-term recursion relation, which is simpler to analyse, following Gardiner [14] we make the substitution

$$\Delta_n = T_{n+1} - T_n. \tag{2.5}$$

This yields the equation $w^+_n\Delta_n - w^-_n\Delta_{n-1} = -1$, and results in $\Delta_L = -T_L$ and $\Delta_{-(M+1)} = T_{-M}$. Assuming that $w^+_n \neq 0$ for all $n \in D$, from these conditions we immediately obtain

$$\Delta_n = \frac{w^-_n}{w^+_n}\Delta_{n-1} - \frac{1}{w^+_n} \tag{2.6}$$

$$\sum_{n\in D} \Delta_n = -T_{-M}. \tag{2.7}$$

Starting from site $-M$ and applying recursively (2.6), we obtain

$$\Delta_n = \prod_{j=-M}^{n} \frac{w^-_j}{w^+_j} T_{-M} - \frac{1}{w^+_n} - \frac{1}{w^+_n} \sum_{i=-M}^{n-1} \prod_{j=i}^{n-1} \frac{w^-_{j+1}}{w^+_j}. \tag{2.8}$$

Note that the last term only runs for $n \geqslant -M+1$. Using (2.8) in (2.7), immediately results in

$$T_{-M}\left(1 + \sum_{k=-M}^{L} \prod_{j=-M}^{k} \frac{w^-_j}{w^+_j}\right) = \sum_{k=-M}^{L} \frac{1}{w^+_k} + \sum_{k=-M+1}^{L} \frac{1}{w^+_k} \sum_{i=-M}^{k-1} \prod_{j=i}^{k-1} \frac{w^-_{j+1}}{w^+_j}. \tag{2.9}$$

Finally, writing $T_n = T_{-M} + \sum_{k=-M}^{n-1} \Delta_k$ and substituting in according to equations (2.8) and (2.9), we get

$$T_n = \frac{1 + \sum_{k=-M}^{n-1} \prod_{j=-M}^{k} \frac{w_j^-}{w_j^+}}{1 + \sum_{k=-M}^{L} \prod_{j=-M}^{k} \frac{w_j^-}{w_j^+}} \left(\sum_{k=-M}^{L} \frac{1}{w_k^+} + \sum_{k=-M+1}^{L} \frac{1}{w_k^+} \sum_{i=-M}^{k-1} \prod_{j=i}^{k-1} \frac{w_{j+1}^-}{w_j^+} \right) - \left(\sum_{k=-M}^{n-1} \frac{1}{w_k^+} + \sum_{k=-M+1}^{n-1} \frac{1}{w_k^+} \sum_{i=-M}^{k-1} \prod_{j=i}^{k-1} \frac{w_{j+1}^-}{w_j^+} \right). \tag{2.10}$$

This expression can be additionally recast as

$$T_n = \frac{1 + \sum_{k=-M}^{n-1} \prod_{j=-M}^{k} \frac{w_j^-}{w_j^+}}{1 + \sum_{k=-M}^{L} \prod_{j=-M}^{k} \frac{w_j^-}{w_j^+}} \left(\sum_{k=-M}^{L} \frac{1}{w_k^+} + \sum_{k=-M}^{L-1} \frac{1}{w_k^+} \sum_{i=k+1}^{L} \prod_{j=k+1}^{i} \frac{w_j^-}{w_j^+} \right) - \left(\sum_{k=-M}^{n-1} \frac{1}{w_k^+} + \sum_{k=-M}^{n-2} \frac{1}{w_k^+} \sum_{i=k+1}^{n-1} \prod_{j=k+1}^{i} \frac{w_j^-}{w_j^+} \right) \tag{2.11}$$

for $n \in D$, and where the sums whose upper limit is $n - \alpha$ run only for $n \geqslant -M + \alpha$. The above equation for the MFPT is an exact expression for quenched disorder, given that it contains explicitly the full dependence on the basic jump transitions $\{w_j^{\pm}\}$.

Let us substitute, in (2.11), $w_k^+ = w_k^- = w_k$ for all k. This corresponds to a symmetrical random walk and we obtain

$$T_n = \frac{n + M + 1}{L + M + 2} \left(\sum_{k=-M}^{L} \frac{1}{w_k} + \sum_{k=-M}^{L-1} \frac{L - k}{w_k} \right) - \left(\sum_{k=-M}^{n-1} \frac{1}{w_k} + \sum_{k=-M}^{n-2} \frac{n - k - 1}{w_k} \right). \tag{2.12}$$

2.2. *Mean residence time*

In the residence time problem, the walker jumps on the unbounded chain, but we compute the probability of finding the walker in the finite region D, as shown in figure 2. Particularly, we are concerned with the mean time that the walker spends in D. From the Laplace transform of the evolution equation (2.1) and taking the limit of equation (2.3*b*), we obtain the corresponding equation for the MRT

$$w_n^+(\tau_{n+1} - \tau_n) + w_n^-(\tau_{n-1} - \tau_n) = -R_n(t = 0) \tag{2.13}$$

where $R_n(t = 0) = 1$ if $n \in D$, or 0 otherwise. For this problem, we assume the presence of a global bias that points to the right, i.e. $w_j^- / w_j^+ < 1$ for all sites j. Thus, equation (2.13) must be supplemented with the condition: $\lim_{n\to+\infty} \tau_n = 0$. Equation (2.13) is also a three-term recursion formula. Therefore, we again make the substitution

$$\Gamma_n = \tau_{n+1} - \tau_n. \tag{2.14}$$

This yields the equations

$$\Gamma_{n-1} = \frac{w_n^+}{w_n^-} \Gamma_n \qquad \text{for } n < -M \tag{2.15a}$$

$$\Gamma_n = \frac{w_n^-}{w_n^+} \Gamma_{n-1} \qquad \text{for } n > L \tag{2.15b}$$

$$\Gamma_n = \frac{w_n^-}{w_n^+} \Gamma_{n-1} - \frac{1}{w_n^+} \qquad \text{for } -M \leqslant n \leqslant L. \tag{2.15c}$$

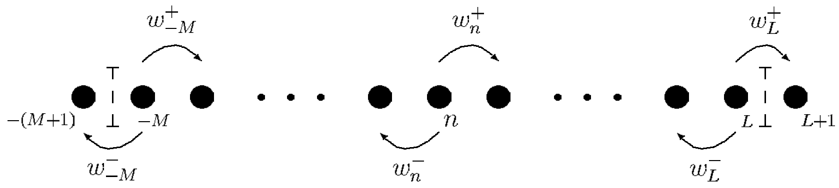

Figure 2. Schematic representation of the residence time problem. The region of interest is the segment between the vertical dashed lines.

We are considering residence in a finite region, then we must take the boundary conditions: $\lim_{n\to\pm\infty}\Gamma_n = 0$.

Applying recursively equation (2.15a), we get

$$\Gamma_k = \prod_{j=k+1}^{-(M+1)} \frac{w_j^+}{w_j^-}\Gamma_{-(M+1)} \qquad \text{for } k \leqslant -(M+2). \tag{2.16}$$

Given that we have assumed $w_j^+\big/w_j^- > 1$, the condition $\lim_{k\to-\infty}\Gamma_k = 0$, imposes that $\Gamma_{-(M+1)} = 0$. Therefore, we immediately obtain $\Gamma_k = 0$ for $k \leqslant -(M+1)$. Thus, from equation (2.14) we get

$$\tau_k = \tau_{-M} \qquad \text{for } k < -M. \tag{2.17}$$

In a similar way, from equation (2.15b), we get

$$\Gamma_k = \prod_{j=L+1}^{k} \frac{w_j^-}{w_j^+}\Gamma_L \qquad \text{for } k \geqslant L+1 \tag{2.18}$$

and the assumption $w_j^-/w_j^+ < 1$ guarantees that $\lim_{k\to+\infty}\Gamma_k = 0$, for any arbitrary Γ_L. On the other hand, from equation (2.15c), using that $\Gamma_{-(M+1)} = 0$, we obtain

$$\Gamma_{-M} = -\frac{1}{w_{-M}^+} \tag{2.19a}$$

$$\begin{aligned}\Gamma_k &= -\frac{1}{w_k^+}\left(1 + \sum_{i=-M}^{k-1}\prod_{j=i}^{k-1}\frac{w_{j+1}^-}{w_j^+}\right)\\ &= -\frac{1}{w_k^+} - \sum_{i=-M}^{k-1}\frac{1}{w_i^+}\prod_{j=i+1}^{k}\frac{w_j^-}{w_j^+} \qquad \text{for } -M < k \leqslant L.\end{aligned} \tag{2.19b}$$

The boundary condition for the MRT when the bias points to the right, $\lim_{n\to+\infty}\tau_n = 0$, and equation (2.14), allows us to write $\tau_n = -\sum_{k=n}^{\infty}\Gamma_k$, and using equations (2.18) and (2.19b) for $k = L$, we obtain

$$\tau_n = \left(\frac{1}{w_L^+} + \sum_{i=-M}^{L-1}\frac{1}{w_i^+}\prod_{j=i+1}^{L}\frac{w_j^-}{w_j^+}\right)\sum_{k=n}^{\infty}\prod_{j=L+1}^{k}\frac{w_j^-}{w_j^+} \qquad \text{for } n > L \tag{2.20}$$

and

$$\tau_L = \left(\frac{1}{w_L^+} + \sum_{i=-M}^{L-1}\frac{1}{w_i^+}\prod_{j=i+1}^{L}\frac{w_j^-}{w_j^+}\right)\left(1 + \sum_{k=L+1}^{\infty}\prod_{j=L+1}^{k}\frac{w_j^-}{w_j^+}\right). \tag{2.21}$$

From equation (2.14), we can also write $\tau_{-M} = \tau_L - \sum_{k=-M}^{L-1} \Gamma_k$, and using equation (2.19*b*), we get, after introducing a change in the order of the sums,

$$\tau_{-M} = \tau_L + \sum_{k=-M}^{L-1} \frac{1}{w_k^+} + \sum_{k=-M}^{L-2} \frac{1}{w_k^+} \sum_{i=k+1}^{L-1} \prod_{j=k+1}^{i} \frac{w_j^-}{w_j^+}. \tag{2.22}$$

Finally, writing $\tau_n = \tau_{-M} + \sum_{k=-M}^{n-1} \Gamma_k$, using equation (2.19*b*), and doing the same change in the order of the sums, we obtain

$$\tau_n = \tau_{-M} - \sum_{k=-M}^{n-1} \frac{1}{w_k^+} - \sum_{k=-M}^{n-2} \frac{1}{w_k^+} \sum_{i=k+1}^{n-1} \prod_{j=k+1}^{i} \frac{w_j^-}{w_j^+} \qquad \text{for } -M < n < L. \tag{2.23}$$

Equations (2.17) and (2.20)–(2.23) are the exact expressions for the MRT for quenched disorder, with full dependence on the set of transitions $\{w_j^\pm\}$.

For the unbiased random walk, the residence problem is not defined, as can be seen from equations (2.20)–(2.21), where the series diverges in the symmetrical case, i.e. $w_j^+ = w_j^-$.

3. Homogeneous chain

The homogeneous biased chain corresponds to the case $w_j^+ = a, w_j^- = b$. Introducing the bias parameter $\gamma = b/a$, equation (2.11) can be easily written as

$$T_n = \frac{L+1-n}{a(1-\gamma)} - \frac{L+M+2}{a(1-\gamma)} \frac{\gamma^n - \gamma^{L+1}}{\gamma^{-(M+1)} - \gamma^{L+1}} \qquad \text{for } -M \leqslant n \leqslant L. \tag{3.1}$$

The study of the drift and diffusive regimes of the MFPT follows from equation (3.1). Additional information about the survival probability in the homogeneous chain was reported in [22, 24].

For the MRT in the homogeneous chain, we get

$$\tau_n = \frac{1}{a} \begin{cases} \dfrac{L+M+1}{1-\gamma} & \text{for } n < -M \\ \dfrac{L-n}{1-\gamma} + \dfrac{1-\gamma^{n+M+1}}{(1-\gamma)^2} & \text{for } -M \leqslant n \leqslant L \\ \dfrac{1-\gamma^{L+M+1}}{(1-\gamma)^2}\gamma^{n-L} & \text{for } n > L \end{cases} \tag{3.2}$$

where $0 < \gamma < 1$. An expression for the residence probability in the homogeneous chain and the limit regimes of equation (3.2) were given in [24].

4. Weak biased disordered chain

Two prototype problems are defined in disordered one-dimensional systems. The first one, the bond disorder or random-barrier problem, corresponds to the situation in which the transfer rates associated with the bond between two neighbour sites are symmetric, $w_n^+ = w_{n+1}^-$. In this kind of problem, the statistical properties of the system are given by the distribution of the random variable which describes the transition rate in each bond. The second class is the site disorder or random-trap problem. Here, the hopping probabilities $w_n^\pm$ associated with each site are symmetric, $w_n^+ = w_{n+1}^-$ [21], and the distribution of hopping rates in each site characterizes the model of disorder. Physical realizations corresponding to each class of problems were summarized by Alexander *et al* [25].

In what follows, we consider site disordered chains. We assume that the hopping probabilities $w_n^{\pm}$ are strictly positive random variables, chosen independently from site to site and identically distributed. However, we are particularly interested in the effects of bias in the chain by external fields. Therefore, we admit that the site transition probabilities are not necessarily symmetric in the sense that $w_n^+ \neq w_n^-$. Thus, our first random-site model is defined by $w_j^+ = a + \xi_j$, $w_j^- = b + \xi_j$, where a and b are positive constants, and $\{\xi_j\}$ are taken as independent but identically distributed random variables with $\langle \xi_j \rangle = 0$. This form of jump transition involves an ordered biased background with an added random medium. The strength of the bias is given by the ratio between a and b and the disorder is characterized by the distribution of variables $\{\xi_j\}$. We introduce the parameter ϵ for bias strength by $b/a = 1 - \epsilon$. This selection of parameters allows us to focus our attention on the small bias limit and to study the transition to the symmetric diffusive behaviour. For practical reasons, we can alternatively introduce this *additive* model by the constraint $w_j^- = w_j^+ - a\epsilon$, where the random variables $\{w_j^+\}$ are taken independent and identically distributed. Additionally, $w_j^+ > a\epsilon$ for $\epsilon > 0$, otherwise w_j^+ is strictly positive for $\epsilon < 0$. Thus, we can write

$$\frac{w_j^-}{w_j^+} = 1 - \frac{a\epsilon}{w_j^+}. \tag{4.1}$$

We will find it useful to define the quantities $\beta_k \equiv \left\langle \left(1/w_j^+\right)^k \right\rangle$, which we assume finite for all $k \geqslant 1$. This class of disorder is known as weak disorder [21–24]. We also define a measure of the fluctuation of the disorder by $\mathcal{F} \equiv \left(\beta_2 - \beta_1^2\right)/\beta_1^2$. In this manner, we can write

$$\left\langle \frac{1}{w_j^+ w_k^+} \right\rangle = \beta_1^2 (1 + \mathcal{F}\delta_{jk}). \tag{4.2}$$

In the next step, we will average over the realizations of disorder the corresponding expressions for MFPT and MRT, up to first order in the bias parameter ϵ. For this model, in both problems we need to use the following expansions:

$$\prod_{j=\alpha}^{\beta} \frac{w_j^-}{w_j^+} \simeq 1 - a\epsilon \sum_{j=\alpha}^{\beta} \frac{1}{w_j^+} \tag{4.3a}$$

$$\sum_{k=\alpha}^{x} \prod_{j=\alpha}^{\beta} \frac{w_j^-}{w_j^+} \simeq x - \alpha + 1 - a\epsilon \sum_{k=\alpha}^{x} \sum_{j=\alpha}^{\beta} \frac{1}{w_j^+} \tag{4.3b}$$

$$\sum_{k=-M}^{x-1} \frac{1}{w_k^+} \prod_{j=k+1}^{x} \frac{w_j^-}{w_j^+} \simeq \sum_{k=-M}^{x-1} \frac{1}{w_k^+} - a\epsilon \sum_{k=-M}^{x-1} \sum_{j=k+1}^{x} \frac{1}{w_k^+ w_j^+} \tag{4.3c}$$

$$\sum_{k=-M}^{x-1} \frac{1}{w_k^+} \sum_{k=k+1}^{x} \prod_{j=k+1}^{i} \frac{w_j^-}{w_j^+} \simeq \sum_{k=-M}^{x-1} \frac{x-k}{w_k^+} - a\epsilon \sum_{k=-M}^{x-1} \sum_{i=k+1}^{x} \sum_{j=k+1}^{i} \frac{1}{w_k^+ w_j^+}. \tag{4.3d}$$

We must note that in the last term of equations (4.3c) and (4.3d) $j \neq k$. Therefore, the averages of these expressions do not involve the fluctuation of disorder. However, when we average equation (2.11), expanding up to first order in ϵ and using equations (4.3a)–(4.3d), we obtain

$$\langle T_n \rangle \simeq \Theta_0(n, L, M)\beta_1 + [\Theta_1(n, L, M) + \Theta_2(n, L, M)\mathcal{F}]\, a\beta_1^2 \epsilon \tag{4.4}$$

where the functions $\Theta_0(n, L, M)$, $\Theta_1(n, L, M)$ and $\Theta_2(n, L, M)$ are defined in the appendix. Here, the fluctuation of disorder $\mathcal{F}$ is introduced by the expansion of the denominator of expression (2.11). Following the calculations given in the appendix, we obtain

$$\Theta_0(n, L, M) = \frac{(L+1-n)(M+1+n)}{2} \tag{4.5a}$$

$$\frac{\Theta_1(n, L, M)}{\Theta_0(n, L, M)} = \frac{L-M+3-2n}{6} \tag{4.5b}$$

$$\frac{\Theta_2(n, L, M)}{\Theta_0(n, L, M)} = \frac{2L+M+3-n}{3(L+M+2)}. \tag{4.5c}$$

Therefore, the averaged MFPT results in

$$\langle T_n \rangle \simeq \frac{(L+1-n)(M+1+n)}{2\beta_1^{-1}} \left[1 + \left(\frac{L-M+3-2n}{6} + \frac{2L+M+3-n}{3(L+M+2)} \mathcal{F} \right) a\beta_1\epsilon \right]. \tag{4.6}$$

Taking the limit $\epsilon \to 0$ in equation (4.6), we arrive at the known expression for the MFPT of a homogeneous symmetrical chain with the effective coefficient β_1^{-1}. Note that, for this class of disorder, the asymmetry in the hopping transitions links the strength of the bias with the fluctuation of disorder.

On the other hand, for the residence problem we need to evaluate the expression

$$\left\langle \sum_{k=n}^{\infty} \prod_{j=L+1}^{k} \frac{w_j^-}{w_j^+} \right\rangle \simeq \sum_{i=0}^{\infty} \exp[-a\epsilon\beta_1(i+n-L)] \simeq \frac{1-a\epsilon\beta_1(n-L)}{a\epsilon\beta_1} \tag{4.7}$$

which is valid for $n \geqslant L+1$. Then, taking $0 < \epsilon \leqslant 1$, so that the bias field points to the right, and using equations (4.3*c*), (4.3*d*) and (4.7), from equations (2.20)–(2.23) we obtain the averaged MRT

$$\langle \tau_n \rangle \simeq \frac{L+M+1}{a\epsilon} \begin{cases} 1 & \text{for } n < -M \\ 1 - \dfrac{(n+M)(n+M+1)}{2(L+M+1)} a\beta_1\epsilon & \text{for } -M \leqslant n \leqslant L \\ 1 - [n-(L-M)/2]\, a\beta_1\epsilon & \text{for } n > L. \end{cases} \tag{4.8}$$

This expression is equivalent to the diffusive regime of a homogeneous chain. Therefore, it can be obtained from equation (3.2) taking $\gamma = 1-\epsilon$ and expanding up to first order in ϵ. The effect of disorder appears as a renormalization of the parameters a and ϵ according to $a \to \beta_1^{-1}$ and $\epsilon \to a\beta_1\epsilon$ [24]. Note that, in the limit $\epsilon \to 0$, from equation (4.8) we obtain a divergence. This is a direct consequence of the fact that the MRT is not a defined quantity for unbiased chains.

We remark that the results given by equations (4.6) and (4.8) are valid for weak disorder. When the quantities β_k are not finite, we have to deal with a perturbative approach to analyse the behaviour of the averaged MFPT and MRT [24].

5. Multiplicative model

Our second asymmetric random-site model is defined by $w_j^+ = a\eta_j$ and $w_j^- = b\eta_j$, where a and b are positive constants and $\{\eta_j\}$ are taken independent but identically distributed positive random variables with $\langle \eta_j \rangle = 1$. Thus, we obtain $w_j^- \big/ w_j^+ = b/a = \gamma$. It can immediately be recognized that the averaged expressions for MFPT and MRT are obtained replacing $1\big/w_k^+$ by $\beta_1 \equiv \langle 1/(a\eta_j) \rangle$ in all the terms of equations (2.11) and (2.20)–(2.23). Therefore, the averaged formulae for this model of disorder correspond to the homogeneous case, given in section 3, with the substitution $a \to \beta_1^{-1}$. We must stress that the averaged expressions obtained in this way are exact expressions for all values of the bias parameter γ. Particularly, taking $\gamma = 1-\epsilon$, up to first order in ϵ, we obtain for the averaged MFPT

$$\langle T_n \rangle \simeq \frac{(L+1-n)(M+1+n)}{2\beta_1^{-1}} \left[1 + \frac{L-M+3-2n}{6}\epsilon\right]. \tag{5.1}$$

As a notable remark, we observe that there is no coupling between the bias ϵ and the fluctuation of disorder $\mathcal{F}$ for the multiplicative model. We emphasize that the multiplicative asymmetric disordered model cannot be easily worked out using FEMA [21, 22] or in general using a perturbative method [24].

6. One reflecting boundary

In this section, we consider a reflecting boundary condition at the left extreme of the interval. The reflecting boundary can be modelled by setting $w^-_{-M} = w^+_{-(M+1)} = 0$. Without loss of generality, we take $M = 0$, therefore our interval of interest is $[0, L]$. From equation (2.11), taking $M = 0$ and $w^-_0 = 0$, we immediately obtain for the MFPT

$$T_0 = \sum_{k=0}^{L} \frac{1}{w^+_k} + \sum_{k=0}^{L-1} \frac{1}{w^+_k} \sum_{i=k+1}^{L} \prod_{j=k+1}^{i} \frac{w^-_j}{w^+_j} \tag{6.1a}$$

$$T_1 = T_0 - \frac{1}{w^+_0} \tag{6.1b}$$

$$T_n = T_0 - \sum_{k=0}^{n-1} \frac{1}{w^+_k} - \sum_{k=0}^{n-2} \frac{1}{w^+_k} \sum_{i=k+1}^{n-1} \prod_{j=k+1}^{i} \frac{w^-_j}{w^+_j} \qquad \text{for } 2 \leqslant n \leqslant L. \tag{6.1c}$$

Equation (6.1a) was reported for discrete time random walks by Murthy and Kehr [13]. The main effect of the reflecting boundary is the disappearance of the denominator in the expression of the MFPT. For the homogeneous (nondisordered) case, for which $w^+_j = a$, $w^-_j = b$ and $\gamma = b/a$, we obtain for $0 \leqslant n \leqslant L$ that

$$T_n = \frac{1}{a(1-\gamma)} \left(L + 1 - n - \frac{\gamma^{n+1} - \gamma^{L+2}}{1-\gamma}\right). \tag{6.2}$$

The small bias expansion ($\gamma = 1 - \epsilon$) of equation (6.2) is given by

$$T_n \simeq \frac{(L+1)(L+2) - n(n+1)}{2a} \left[1 - \left(\frac{L}{3} + \frac{n(n+1)}{3(L+2+n)}\right)\epsilon\right]. \tag{6.3}$$

In the guideline of section 4, we can compute the averaged MFPT for a walker in a chain with reflecting–absorbing boundary conditions and additive disorder. In this manner, we obtain up to first order in ϵ

$$\langle T_n \rangle \simeq \frac{(L+1)(L+2) - n(n+1)}{2\beta_1^{-1}} \left[1 - \left(\frac{L}{3} + \frac{n(n+1)}{3(L+2+n)}\right) a\beta_1\epsilon\right]. \tag{6.4}$$

Equation (6.4) is exactly the same as (6.3) for weak biased nondisordered chains if we set $a \to \beta_1^{-1}$ and $\epsilon \to a\beta_1\epsilon$. Strikingly, for reflecting–absorbing boundaries the fluctuation of disorder is not present in the averaged MFPT. From equations (6.2) and (6.3), the same argument presented in section 5 allows us to reckon the averaged MFPT for a disordered chain under multiplicative disorder, substituting a by β_1^{-1}.

The presence of a reflecting boundary at the left of the interval has no effect on the RT problem when the bias points to the right. This fact can be easily seen from equations (2.20)–(2.23), where we found that MRT expressions do not depend on the hopping transitions w^-_{-M} and $w^+_{-(M+1)}$. However, if the reflecting boundary is at the right of the interval, when the bias points to the right, the MRT diverges as expected. This fact can be seen from equations (2.21)–(2.23) taking $w^-_{L+1} = 0$ and the limit $w^+_L \to 0$ (see figure 2).

7. Concluding remarks

We have presented an algebraic method for calculating MFPT and MRT of a random walk on one-dimensional lattices for quenched disorder. The starting points are the one step equations (2.4) and (2.13). We have obtained the exact solution (2.11) for the MFPT and the exact expressions (2.17) and (2.20)–(2.23) for the MRT. Also, we have considered a reflecting boundary in section 6. Two models for site disorder in the chain were considered, namely, additive and multiplicative. The expressions for MFPT and MRT were exactly averaged for both kinds of disorder. The main difference between these models is in the coupling of the inverse moments of jump transitions and the bias parameter in the averaged quantities. For the additive model of section 4, the terms to first order in the bias parameter are proportional to $a\beta_1\epsilon$ (see equations (4.6), (4.8) and (6.4)). Moreover, for absorbing–absorbing boundaries, the bias links the strength parameter ϵ with the fluctuation of disorder $\mathcal{F}$ in the averaged MFPT (see equation (4.6)). On the other hand, for the multiplicative model, the terms to first order in the bias parameter are proportional to $a\epsilon$ and there is no kind of link between ϵ and $\mathcal{F}$ in the averaged MFPT.

In this work we have considered models of site disorder. The pair of transition probabilities w^+_j and w^-_j, associated with the site j, has the same distribution through the random variable ξ_j or η_j. In addition, the exact equation (2.10) allows us to consider models of bond disorder. In this case, the pair of jump probabilities w^+_j and w^-_{j+1}, associated with the bond between the sites j and $j+1$, has the same distribution. To tackle this problem, we only need to rewrite the sums in the prefactor in the following manner:

$$\sum_{k=-M}^{x} \prod_{j=-M}^{k} \frac{w^-_j}{w^+_j} = w^-_{-M} \left(\frac{1}{w^+_{-M}} + \sum_{k=-M}^{x-1} \frac{1}{w^+_{k+1}} \prod_{j=-M}^{k} \frac{w^-_{j+1}}{w^+_j} \right). \qquad (7.1)$$

Similar arrangement can be made with the residence time expressions to deal with bond disordered problems.

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Appendix

In order to perform the average over disorder for the additive model of section 4, we first need to evaluate the following functions:

$$f_0(x, M) \equiv \sum_{k=-M}^{x-1} (x-k) = \frac{(x+M)(x+M+1)}{2} \qquad (A.1)$$

$$f_1(x, M) \equiv \sum_{k=-M}^{x-1} \sum_{i=k+1}^{x} \sum_{j=k+1}^{i} 1 = \frac{(x+M)(x+M+1)(x+M+2)}{6} \tag{A.2}$$

$$f_2(x, M) \equiv \sum_{k=-M}^{x-1} \sum_{i=k+1}^{x} \sum_{j=k+1}^{i} \delta_{jk} = 0 \tag{A.3}$$

$$f_3(x, L, M) \equiv \sum_{k=-M}^{L} \sum_{i=-M}^{x} \sum_{j=-M}^{i} 1 = \frac{(L+M+1)(x+M+1)(x+M+2)}{2} \tag{A.4}$$

$$f_4(x, L, M) \equiv \sum_{k=-M}^{L-1} \sum_{i=-M}^{x} \sum_{j=-M}^{i} (L-k) = \frac{(L+M)(L+M+1)(x+M+1)(x+M+2)}{4} \tag{A.5}$$

$$f_5(x, L, M) \equiv \sum_{k=-M}^{L} \sum_{i=-M}^{x} \sum_{j=-M}^{i} \delta_{jk} = \frac{(x+M+1)(x+M+2)}{2} \tag{A.6}$$

$$f_6(x, L, M) \equiv \sum_{k=-M}^{L-1} \sum_{i=-M}^{x} \sum_{j=-M}^{i} (L-k)\delta_{jk} = \frac{(3L+2M-x)(x+M+1)(x+M+2)}{6}. \tag{A.7}$$

Using the functions defined above, the functions $\Theta_0(n, L, M)$, $\Theta_1(n, L, M)$ and $\Theta_2(n, L, M)$, that appear in equation (4.4), are defined by

$$\Theta_0(n, L, M) \equiv \frac{n+M+1}{L+M+2}\,[(L+M+1) + f_0(L, M)] - [(n+M) + f_0(n-1, M)] \tag{A.8}$$

$$\Theta_1(n, L, M) \equiv \frac{n+M+1}{L+M+2}\left[\frac{1}{L+M+2}\,(f_3(L, L, M) + f_4(L, L, M)) - \frac{1}{n+M+1}\right.$$
$$\left.\times\,(f_3(n-1, L, M) + f_4(n-1, L, M)) - f_1(L, M)\right] + f_1(n-1, M) \tag{A.9}$$

$$\Theta_2(n, L, M) \equiv \frac{n+M+1}{L+M+2}\left[\frac{1}{L+M+2}\,(f_5(L, L, M) + f_6(L, L, M))\right.$$
$$\left. - \frac{1}{n+M+1}\,(f_5(n-1, L, M) + f_6(n-1, L, M))\right]. \tag{A.10}$$

Equations (4.5*a*)–(4.5*c*) follow from equations (A.8)–(A.10), respectively, replacing the functions f_i by their explicit expressions.

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